

# BIVARIATE CHENEY-SHARMA OPERATORS ON SIMPLEX

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**ABSTRACT.** In this paper, we consider bivariate Cheney-Sharma operators which are not the tensor product construction. Precisely, we show that these operators preserve Lipschitz condition of a given Lipschitz continuous function  $f$  and also the properties of the modulus of continuity function when  $f$  is a modulus of continuity function.

## 1. Introduction

The most celebrated linear positive operators for the uniform approximation of continuous real valued functions on  $[0, 1]$  are Bernstein polynomials. As it is well known, besides approximation results, Bernstein polynomials have some nice retaining properties. The most referred study in this direction was due to Brown, Elliott and Paget [7] where they gave an elementary proof for the preservation of the Lipschitz constant and order of a Lipschitz continuous function by the Bernstein polynomials. Whereas, Lindvall previously obtained this result in terms of probabilistic methods in [20]. Moreover, in [19] Li proved that Bernstein polynomials also preserve the properties of the function of modulus of continuity. The same problems for some other type univariate or multivariate linear positive operators were solved by either using an elementary or probabilistic way (see, e.g. [3]-[6], [8], [9], [14], [15], [17], [18], [28]).

In Abel-Jensen identity (see [2], p.326)

$$(u + v)(u + v + m\beta)^{m-1} = \sum_{k=0}^m \binom{m}{k} u(u + k\beta)^{k-1} v[v + (m - k)\beta]^{m-k-1} \quad (1.1)$$

where  $u, v$  and  $\beta \in \mathbb{R}$ , by taking  $u = x$ ,  $v = 1 - x$  and  $m = n$ , Cheney-Sharma [11] introduced the following Bernstein type operators for  $f \in C[0, 1]$ ,  $x \in [0, 1]$  and  $n \in \mathbb{N}$

$$Q_n^\beta(f; x) = (1 + n\beta)^{1-n} \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x(x + k\beta)^{k-1} (1 - x)[1 - x + (n - k)\beta]^{n-k-1},$$

where  $\beta$  is a nonnegative real parameter. For these operators, tensor product of them and their some generalizations we can cite the papers [1], [10], [12], [21]-[27] and the monograph [2]. Remark that from [11] and [21] we know that  $Q_n^\beta$  operators reproduce constant functions and linear functions. Very recently, in [6] the authors showed that univariate Cheney-Sharma operators preserve the Lipschitz constant and order of a Lipschitz continuous function as well as the properties of the function of modulus of continuity.

We now introduce the notations, some needful definitions and the construction of the bivariate operators.

Throughout the paper, we shall use the standard notations given below.

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Let  $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ ,  $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0^2$ ,  $\mathbf{e} = (1, 1)$ ,  $\mathbf{0} = (0, 0)$ ,  $0 \leq \beta \in \mathbb{R}$  and  $n \in \mathbb{N}$ . We denote as usual

$$|\mathbf{x}| := x_1 + x_2, \quad \mathbf{x}^{\mathbf{k}} := x_1^{k_1} x_2^{k_2}, \quad |\mathbf{k}| := k_1 + k_2, \quad \mathbf{k}! := k_1! k_2!, \quad \beta \mathbf{x} = (\beta x_1, \beta x_2)$$

and

$$\binom{n}{\mathbf{k}} := \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!}, \quad \sum_{|\mathbf{k}| \leq n} := \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1}.$$

We also denote the two dimensional simplex by

$$S := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, |\mathbf{x}| \leq 1\}.$$

Moreover,  $\mathbf{x} \leq \mathbf{y}$  stands for  $x_i \leq y_i$ ,  $i = 1, 2$ .

We now construct the non-tensor product Cheney-Sharma operators. From (1.1) it is clear that

$$(1 + n\beta)^{n-1} = \sum_{k_1=0}^n \binom{n}{k_1} x_1 (x_1 + k_1\beta)^{k_1-1} (1 - x_1) [1 - x_1 + (n - k_1)\beta]^{n-k_1-1}.$$

In (1.1), taking  $u = x_2$ ,  $v = 1 - x_1 - x_2$  and  $m = n - k_1$  we have

$$\begin{aligned} (1 - x_1) [1 - x_1 + (n - k_1)\beta]^{n-k_1-1} &= \sum_{k_2=0}^{n-k_1} \binom{n-k_1}{k_2} x_2 (x_2 + k_2\beta)^{k_2-1} (1 - x_1 - x_2) \\ &\quad \times [1 - x_1 - x_2 + (n - k_1 - k_2)\beta]^{n-k_1-k_2-1}. \end{aligned}$$

Using this result in the above equality we find that

$$\begin{aligned} (1 + n\beta)^{n-1} &= \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} x_1 x_2 (x_1 + k_1\beta)^{k_1-1} (x_2 + k_2\beta)^{k_2-1} \\ &\quad \times (1 - x_1 - x_2) [1 - x_1 - x_2 + (n - k_1 - k_2)\beta]^{n-k_1-k_2-1} \\ &= \sum_{|\mathbf{k}| \leq n} \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}. \end{aligned}$$

or

$$1 = (1 + n\beta)^{1-n} \sum_{|\mathbf{k}| \leq n} \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}.$$

In this paper, for a continuous real valued function  $f$  defined on  $S$  we consider the non-tensor product bivariate extension of the operators  $Q_n^\beta(f; x)$  defined by

$$\begin{aligned} G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{|\mathbf{k}| \leq n} f\left(\frac{\mathbf{k}}{n}\right) \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} \\ &\quad \times (1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1} \end{aligned} \quad (1.2)$$

where  $\beta$  is a nonnegative real parameter,  $\mathbf{x} \in S$  and  $n \in \mathbb{N}$ . We observe that for  $\beta = 0$  these operators reduce to non-tensor product bivariate Bernstein polynomials (see [13],[16]).

**Definition 1.** (see, e.g.[9]) A continuous real valued function  $f$  defined on  $A \subseteq \mathbb{R}^2$  is said to be Lipschitz continuous function of order  $\mu$ ,  $0 < \mu \leq 1$  on  $A$ , if there exists  $M > 0$  such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq M \sum_{i=1}^2 |x_i - y_i|^\mu$$

for all  $\mathbf{x}, \mathbf{y} \in A$ . The set of Lipschitz continuous functions of order  $\mu$  with Lipschitz constant  $M$  on  $A$  is denoted by  $Lip_M(\mu, A)$ .

**Definition 2.** (see, e.g.[8]) If a bivariate nonnegative and continuous function  $\omega(\mathbf{u})$  satisfies the following conditions, then it is called a function of modulus of continuity.

- (a)  $\omega(\mathbf{0}) = 0$ ,
- (b)  $\omega(\mathbf{u})$  is a non-decreasing function in  $\mathbf{u}$ , i.e.,  $\omega(\mathbf{u}) \geq \omega(\mathbf{v})$  for  $\mathbf{u} \geq \mathbf{v}$ ,
- (c)  $\omega(\mathbf{u})$  is semi-additive, i.e.,  $\omega(\mathbf{u} + \mathbf{v}) \leq \omega(\mathbf{u}) + \omega(\mathbf{v})$ .

## 2. Main results

In this section, inspired by the paper of Cao, Ding and Xu [9], including preservation properties of multivariate Baskakov operators, we show that non-tensor product Cheney-Sharma operators defined by  $G_n^\beta$  preserve the Lipschitz condition of a given Lipschitz continuous function  $f$  and properties of the function of modulus of continuity when the attached function  $f$  is a modulus of continuity function.

**Theorem 3.** If  $f \in Lip_M(\mu, S)$ , then  $G_n^\beta(f) \in Lip_M(\mu, S)$  for all  $n \in \mathbb{N}$ .

*Proof.* Let  $\mathbf{x}, \mathbf{y} \in S$  such that  $\mathbf{y} \geq \mathbf{x}$ . From (1.2) we have

$$\begin{aligned} G_n^\beta(f; \mathbf{y}) &= (1 + n\beta)^{1-n} \sum_{i_1=0}^n \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} \mathbf{y}^{\mathbf{e}} (\mathbf{y} + \mathbf{i}\beta)^{\mathbf{i}-\mathbf{e}} \\ &\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{i}|)\beta]^{n-|\mathbf{i}|-1} \\ &= (1 + n\beta)^{1-n} \sum_{i_1=0}^n \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} y_1 (y_1 + i_1\beta)^{i_1-1} \\ &\quad \times y_2 (y_2 + i_2\beta)^{i_2-1} (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{i}|)\beta]^{n-|\mathbf{i}|-1}. \end{aligned}$$

Setting  $u = x_1$ ,  $v = y_1 - x_1$ ,  $m = i_1$  and  $u = x_2$ ,  $v = y_2 - x_2$ ,  $m = i_2$ , respectively, in (1.1), we find

$$y_1 (y_1 + i_1\beta)^{i_1-1} = \sum_{k_1=0}^{i_1} \binom{i_1}{k_1} x_1 (x_1 + k_1\beta)^{k_1-1} (y_1 - x_1) [y_1 - x_1 + (i_1 - k_1)\beta]^{i_1-k_1-1}$$

and

$$y_2 (y_2 + i_2\beta)^{i_2-1} = \sum_{k_2=0}^{i_2} \binom{i_2}{k_2} x_2 (x_2 + k_2\beta)^{k_2-1} (y_2 - x_2) [y_2 - x_2 + (i_2 - k_2)\beta]^{i_2-k_2-1}.$$

Therefore,

$$\begin{aligned}
G_n^\beta(f; \mathbf{y}) &= (1+n\beta)^{1-n} \sum_{i_1=0}^n \sum_{i_2=0}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \binom{n}{\mathbf{i}} \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \binom{i_1}{k_1} \binom{i_2}{k_2} x_1 x_2 \\
&\quad \times (x_1 + k_1\beta)^{k_1-1} (x_2 + k_2\beta)^{k_2-1} (y_1 - x_1)(y_2 - x_2) \\
&\quad \times [y_1 - x_1 + (i_1 - k_1)\beta]^{i_1-k_1-1} [y_2 - x_2 + (i_2 - k_2)\beta]^{i_2-k_2-1} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{i}|)\beta]^{n-|\mathbf{i}|-1} \\
&= (1+n\beta)^{1-n} \sum_{i_1=0}^n \sum_{k_1=0}^{i_1} \sum_{i_2=0}^{n-i_1} \sum_{k_2=0}^{i_2} f\left(\frac{\mathbf{i}}{n}\right) \frac{n!}{\mathbf{k}!(n-|\mathbf{i}|)!(i_1-k_1)!(i_2-k_2)!} \\
&\quad \times \mathbf{x}^{\mathbf{e}}(\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y} - \mathbf{x})^{\mathbf{e}} [\mathbf{y} - \mathbf{x} + (\mathbf{i} - \mathbf{k})\beta]^{\mathbf{i}-\mathbf{k}-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{i}|)\beta]^{n-|\mathbf{i}|-1}.
\end{aligned}$$

Changing the order of the above summations and then letting  $\mathbf{i} - \mathbf{k} = \mathbf{l}$  we obtain

$$\begin{aligned}
G_n^\beta(f; \mathbf{y}) &= (1+n\beta)^{1-n} \sum_{k_1=0}^n \sum_{i_1=k_1}^n \sum_{k_2=0}^{n-i_1} \sum_{i_2=k_2}^{n-i_1} f\left(\frac{\mathbf{i}}{n}\right) \frac{n!}{\mathbf{k}!(n-|\mathbf{i}|)!(i_1-k_1)!(i_2-k_2)!} \\
&\quad \times \mathbf{x}^{\mathbf{e}}(\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y} - \mathbf{x})^{\mathbf{e}} [(\mathbf{y} - \mathbf{x}) + (\mathbf{i} - \mathbf{k})\beta]^{\mathbf{i}-\mathbf{k}-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{i}|)\beta]^{n-|\mathbf{i}|-1} \\
&= (1+n\beta)^{1-n} \sum_{k_1=0}^n \sum_{l_1=0}^{n-k_1} \sum_{k_2=0}^{n-k_1-l_1} \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \\
&\quad \times \mathbf{x}^{\mathbf{e}}(\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}}(\mathbf{y} - \mathbf{x})^{\mathbf{e}}(\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{\mathbf{l}-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned}
G_n^\beta(f; \mathbf{x}) &= (1+n\beta)^{1-n} \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} f\left(\frac{\mathbf{k}}{n}\right) \binom{n}{\mathbf{k}} \mathbf{x}^{\mathbf{e}}(\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1}.
\end{aligned}$$

In (1.1), if we put  $y_1 - x_1$ ,  $1 - y_1 - x_2$  and  $n - |\mathbf{k}|$  in place of  $u$ ,  $v$  and  $m$ , respectively, one has

$$\begin{aligned}
&(1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1} \\
&= \sum_{l_1=0}^{n-|\mathbf{k}|} \binom{n-|\mathbf{k}|}{l_1} (y_1 - x_1)(y_1 - x_1 + l_1\beta)^{l_1-1} (1 - y_1 - x_2) \\
&\quad \times [1 - y_1 - x_2 + (n - |\mathbf{k}| - l_1)\beta]^{n-|\mathbf{k}|-l_1-1}.
\end{aligned}$$

Again in the equality (1.1), we replace  $u, v$  and  $m$  by  $y_2 - x_2, 1 - |\mathbf{y}|$  and  $m = n - |\mathbf{k}| - l_1$ , respectively, we find

$$\begin{aligned} & (1 - y_1 - x_2) [1 - y_1 - x_2 + (n - |\mathbf{k}| - l_1)\beta]^{n-|\mathbf{k}|-l_1-1} \\ &= \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} \binom{n-|\mathbf{k}|-l_1}{l_2} (y_2 - x_2)(y_2 - x_2 + l_2\beta)^{l_2-1} \\ & \quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}. \end{aligned}$$

Making use of this in the above equality leads to

$$\begin{aligned} & (1 - |\mathbf{x}|) [1 - |\mathbf{x}| + (n - |\mathbf{k}|)\beta]^{n-|\mathbf{k}|-1} \\ &= \sum_{l_1=0}^{n-|\mathbf{k}|} \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} \binom{n-|\mathbf{k}|}{l_1} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\ & \quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}. \end{aligned}$$

Thus we conclude that

$$\begin{aligned} G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{k_1=0}^n \sum_{k_2=0}^{n-k_1} \sum_{l_1=0}^{n-|\mathbf{k}|} \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}}{n}\right) \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \\ & \quad \times \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\ & \quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}. \end{aligned}$$

Now changing the order of the two summations in the middle, we obtain

$$\begin{aligned} G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{k_1=0}^n \sum_{l_1=0}^{n-k_1} \sum_{k_2=0}^{n-k_1-l_1} \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} f\left(\frac{\mathbf{k}}{n}\right) \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \\ & \quad \times \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\ & \quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}. \end{aligned} \tag{2.2}$$

So, from (2.1) and (2.2) it follows that

$$\begin{aligned} G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{k_1=0}^n \sum_{l_1=0}^{n-k_1} \sum_{k_2=0}^{n-k_1-l_1} \sum_{l_2=0}^{n-|\mathbf{k}|-l_1} \left[ f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \\ & \quad \times \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\ & \quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}. \end{aligned}$$

Now interchanging the order of the summations two times successively and using the equality

$$\frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} = \binom{n}{\mathbf{l}} \binom{n-|\mathbf{l}|}{k_1} \binom{n-k_1-|\mathbf{l}|}{k_2}$$

we find

$$\begin{aligned}
G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{k_1=0}^{n-l_1} \sum_{l_2=0}^{n-k_1-l_1} \sum_{k_2=0}^{n-k_1-|l|} \left[ f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \\
&\quad \times \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1} \\
&= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \sum_{k_1=0}^{n-|l|} \sum_{k_2=0}^{n-k_1-|l|} \left[ f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \\
&\quad \times \frac{n!}{\mathbf{k}!\mathbf{l}!(n-|\mathbf{k}|-|\mathbf{l}|)!} \mathbf{x}^{\mathbf{e}} (\mathbf{x} + \mathbf{k}\beta)^{\mathbf{k}-\mathbf{e}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1} \\
&= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left[ f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \binom{n}{1} \\
&\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \sum_{k_1=0}^{n-|l|} \binom{n-|l|}{k_1} x_1 (x_1 + k_1\beta)^{k_1-1} \\
&\quad \times \sum_{k_2=0}^{n-k_1-|l|} \binom{n-k_1-|l|}{k_2} x_2 (x_2 + k_2\beta)^{k_2-1} \\
&\quad \times (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}$$

Taking  $u = x_2$ ,  $v = 1 - |\mathbf{y}|$  and  $m = n - k_1 - |\mathbf{l}|$  in (1.1), it is easily seen that

$$\begin{aligned}
& (x_2 + 1 - |\mathbf{y}|) [x_2 + 1 - |\mathbf{y}| + (n - k_1 - |\mathbf{l}|)\beta]^{n-k_1-|\mathbf{l}|-1} \\
&= \sum_{k_2=0}^{n-k_1-|\mathbf{l}|} \binom{n-k_1-|\mathbf{l}|}{k_2} x_2 (x_2 + k_2\beta)^{k_2-1} (1 - |\mathbf{y}|) [1 - |\mathbf{y}| + (n - |\mathbf{k}| - |\mathbf{l}|)\beta]^{n-|\mathbf{k}|-|\mathbf{l}|-1}.
\end{aligned}$$

Hence we can write

$$\begin{aligned}
G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left[ f\left(\frac{\mathbf{k}+\mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \binom{n}{1} \\
&\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} \sum_{k_1=0}^{n-|l|} \binom{n-|l|}{k_1} x_1 (x_1 + k_1\beta)^{k_1-1} \\
&\quad \times (x_2 + 1 - |\mathbf{y}|) [x_2 + 1 - |\mathbf{y}| + (n - k_1 - |\mathbf{l}|)\beta]^{n-k_1-|\mathbf{l}|-1}.
\end{aligned}$$

Again in (1.1), we replace  $u$ ,  $v$  and  $m$  by  $x_1$ ,  $x_2 + 1 - |\mathbf{y}|$  and  $m = n - |\mathbf{l}|$ , respectively, to obtain

$$\begin{aligned}
& (x_1 + x_2 + 1 - |\mathbf{y}|) [x_1 + x_2 + 1 - |\mathbf{y}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\
&= (1 - |\mathbf{y} - \mathbf{x}|) [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\
&= \sum_{k_1=0}^{n-|\mathbf{l}|} \binom{n-|\mathbf{l}|}{k_1} x_1 (x_1 + k_1\beta)^{k_1-1} (x_2 + 1 - |\mathbf{y}|) \\
&\quad \times [x_2 + 1 - |\mathbf{y}| + (n - k_1 - |\mathbf{l}|)\beta]^{n-k_1-|\mathbf{l}|-1}.
\end{aligned}$$

This leads to

$$\begin{aligned}
G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left[ f\left(\frac{\mathbf{k} + \mathbf{l}}{n}\right) - f\left(\frac{\mathbf{k}}{n}\right) \right] \binom{n}{\mathbf{l}} \\
&\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{\mathbf{l} - \mathbf{e}} (1 - |\mathbf{y} - \mathbf{x}|) \\
&\quad \times [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1}.
\end{aligned} \tag{2.3}$$

Since  $f \in Lip_M(\mu, S)$ , we can get

$$\begin{aligned}
\left| G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) \right| &\leq M (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left[ \left(\frac{l_1}{n}\right)^\mu + \left(\frac{l_2}{n}\right)^\mu \right] \binom{n}{\mathbf{l}} \\
&\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{\mathbf{l} - \mathbf{e}} (1 - |\mathbf{y} - \mathbf{x}|) \\
&\quad \times [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\
&= M \left[ G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x}) + G_n^\beta(t_2^\mu; \mathbf{y} - \mathbf{x}) \right].
\end{aligned} \tag{2.4}$$

Now consider the term  $G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x})$ . With the help of the equality  $\binom{n}{\mathbf{l}} = \binom{n}{l_1} \binom{n-l_1}{l_2}$  we can write

$$\begin{aligned}
G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left(\frac{l_1}{n}\right)^\mu \binom{n}{\mathbf{l}} (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{\mathbf{l} - \mathbf{e}} \\
&\quad \times (1 - |\mathbf{y} - \mathbf{x}|) [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\
&= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \left(\frac{l_1}{n}\right)^\mu \binom{n}{l_1} (y_1 - x_1)(y_1 - x_1 + l_1\beta)^{l_1-1} \\
&\quad \times \sum_{l_2=0}^{n-l_1} \binom{n-l_1}{l_2} (y_2 - x_2)(y_2 - x_2 + l_2\beta)^{l_2-1} \\
&\quad \times (1 - |\mathbf{y} - \mathbf{x}|) [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1}.
\end{aligned}$$

In the equality (1.1), if we take  $y_2 - x_2$ ,  $1 - |\mathbf{y} - \mathbf{x}|$  and  $n - l_1$  in place of  $u$ ,  $v$  and  $m$ , respectively, then we get

$$\begin{aligned}
&[1 - (y_1 - x_1)] [1 - (y_1 - x_1) + (n - l_1)\beta]^{n-l_1-1} \\
&= \sum_{l_2=0}^{n-l_1} \binom{n-l_1}{l_2} (y_2 - x_2)(y_2 - x_2 + l_2\beta)^{l_2-1} (1 - |\mathbf{y} - \mathbf{x}|) \\
&\quad \times [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \left(\frac{l_1}{n}\right)^\mu \binom{n}{l_1} (y_1 - x_1)(y_1 - x_1 + l_1\beta)^{l_1-1} \\
&\quad \times [1 - (y_1 - x_1)] [1 - (y_1 - x_1) + (n - l_1)\beta]^{n-l_1-1} \\
&= Q_n^\beta(t_1^\mu; y_1 - x_1).
\end{aligned}$$

Applying the Hölder inequality with conjugate pairs  $p = \frac{1}{\mu}$  and  $q = \frac{1}{1-\mu}$ , we find

$$G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x}) = Q_n^\beta(t_1^\mu; y_1 - x_1) \leq \left[ Q_n^\beta(t_1; y_1 - x_1) \right]^\mu \left[ Q_n^\beta(1; y_1 - x_1) \right]^{1-\mu}.$$

As mentioned before, since the univariate Cheney-Sharma operators given by  $Q_n^\beta$  reproduce constant and linear functions we reach to

$$G_n^\beta(t_1^\mu; \mathbf{y} - \mathbf{x}) \leq (y_1 - x_1)^\mu.$$

Similarly,

$$G_n^\beta(t_2^\mu; \mathbf{y} - \mathbf{x}) \leq (y_2 - x_2)^\mu.$$

Thus from (2.4) it follows that

$$\left| G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) \right| \leq M[(y_1 - x_1)^\mu + (y_2 - x_2)^\mu]$$

which implies that  $G_n^\beta(f) \in Lip_M(\mu, S)$ . In a similar way the same result can be found for  $\mathbf{x} \geq \mathbf{y}$ . If  $x_1 \geq y_1$ ,  $x_2 \leq y_2$ , then we obtain from the above result for  $(y_1, x_2) \in S$  that

$$\begin{aligned} \left| G_n^\beta(f; \mathbf{y}) - G_n^\beta(f; \mathbf{x}) \right| &\leq \left| G_n^\beta(f; (x_1, x_2)) - G_n^\beta(f; (y_1, x_2)) \right| \\ &\quad + \left| G_n^\beta(f; (y_1, y_2)) - G_n^\beta(f; (y_1, x_2)) \right| \\ &\leq M[(y_1 - x_1)^\mu + (y_2 - x_2)^\mu] \end{aligned}$$

Finally, for the case  $x_1 \leq y_1$ ,  $x_2 \geq y_2$  we have the same result. This completes the proof.  $\square$

**Theorem 4.** *If  $\omega$  is a modulus of continuity function, then  $G_n^\beta(\omega)$  is also a modulus of continuity function for all  $n \in \mathbb{N}$ .*

*Proof.* Let  $\mathbf{x}, \mathbf{y} \in S$  such that  $\mathbf{y} \geq \mathbf{x}$ . Regarding  $f$  as a modulus of continuity function  $\omega$  in (2.3) we have

$$\begin{aligned} G_n^\beta(\omega; \mathbf{y}) - G_n^\beta(\omega; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \left[ \omega\left(\frac{\mathbf{k} + \mathbf{l}}{n}\right) - \omega\left(\frac{\mathbf{k}}{n}\right) \right] \binom{n}{\mathbf{l}} \\ &\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} (1 - |\mathbf{y} - \mathbf{x}|) \\ &\quad \times [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \end{aligned}$$

which means that  $G_n^\beta(\omega; \mathbf{y}) \geq G_n^\beta(\omega; \mathbf{x})$  when  $\mathbf{y} \geq \mathbf{x}$ .

Moreover, from the property (c) of modulus of continuity function  $\omega$ , we can write

$$\begin{aligned} G_n^\beta(\omega; \mathbf{y}) - G_n^\beta(\omega; \mathbf{x}) &= (1 + n\beta)^{1-n} \sum_{l_1=0}^n \sum_{l_2=0}^{n-l_1} \omega\left(\frac{\mathbf{l}}{n}\right) \binom{n}{\mathbf{l}} \\ &\quad \times (\mathbf{y} - \mathbf{x})^{\mathbf{e}} (\mathbf{y} - \mathbf{x} + \mathbf{l}\beta)^{1-\mathbf{e}} (1 - |\mathbf{y} - \mathbf{x}|) \\ &\quad \times [1 - |\mathbf{y} - \mathbf{x}| + (n - |\mathbf{l}|)\beta]^{n-|\mathbf{l}|-1} \\ &= G_n^\beta(\omega; \mathbf{y} - \mathbf{x}). \end{aligned}$$

This shows that  $G_n^\beta(\omega)$  is semi-additive. Finally, from the definition of  $G_n^\beta$  it is obvious that  $G_n^\beta(\omega; \mathbf{0}) = \omega(\mathbf{0}) = 0$ . Therefore  $G_n^\beta(\omega)$  itself is a function of modulus of continuity when  $\omega$  is so.  $\square$

## REFERENCES

- [1] O. Agratini, I. A. Rus, *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Carolin., **44**(2003), no.3, 555-563.
- [2] F. Altomare, M. Campiti, *Korovkin-type approximation theory and its applications*, Walter de Gruyter, Berlin-New York, 1994.
- [3] G. Başcanbaz-Tunca, Y. Tuncer, *On a Chlodovsky variant of a multivariate beta operator*, J. Comput. Appl. Math., **235**(2011), no.16, 4816-4824.



- [4] G. Başcanbaz-Tunca, F. Taşdelen, *On Chlodovsky form of the Meyer-König and Zeller operators*, An. Univ. Vest Timiş. Ser. Mat.-Inform., , **49**(2011), no.2, 137-144.
- [5] G. Başcanbaz-Tunca, H. G. İnce-İlarslan, A. Erençin, *Bivariate Bernstein type operators*, Appl. Math. Comput., **273**(2016), 543-552.
- [6] G. Başcanbaz-Tunca, A. Erençin, F. Taşdelen, *Some properties of Bernstein type Cheney and Sharma operators*(submitted).
- [7] B.M. Brown, D. Elliott, D.F. Paget, *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function*, J. Approx. Theory, **49**(1987), no.2, 196-199.
- [8] F. Cao, *Modulus of continuity, K-functional and Stancu operator on a simplex*, Indian J. Pure Appl. Math., **35**(2004), no.12, 1343-1364.
- [9] F. Cao, C. Ding, Z. Xu, *On multivariate Baskakov operator*, J. Math. Anal. Appl., **307**(2005), no.1, 274-291.
- [10] T. Căţinaş, D. Otrocol, *Iterates of multivariate Cheney-Sharma operators*, J. Comput. Anal. Appl., **15**(2013), no.7, 1240-1246.
- [11] E. W. Cheney, A. Sharma, *On a generalization of Bernstein polynomials*, Riv. Mat. Univ. Parma, **2**(5)(1964), 77-84.
- [12] M. Crăciun, *Approximation operators constructed by means of Sheffer sequences*, Rev. Anal. Numér. Théor. Approx., **30**(2001), no.2, 135-150.
- [13] C. Ding, F. Cao, *K-functionals and multivariate Bernstein polynomials*, J. Approx. Theory, **(155)**(2008), no.2, 125-135.
- [14] A. Erençin, G. Başcanbaz-Tunca, F. Taşdelen, *Some preservation properties of MKZ-Stancu type operators*, Sarejevo J. Math., **10**(22)(2014), no.1, 93-102.
- [15] A. Erençin, G. Başcanbaz-Tunca, F. Taşdelen, *Some properties of the operators defined by Lupaş*, Rev. Anal. Numér. Théor. Approx., **43**(2014), no.2, 168-174.
- [16] M. D. Farcas, *About Bernstein polynomials*, An. Univ. Craivo Ser. Mat. Inform., **(35)**(2008), 117-121.
- [17] M. K. Khan, *Approximation properties of Beta operators*, Progress in approximation theory, Academic Press, Boston, MA, (1991), 483-495.
- [18] M. K. Khan, M. A. Peters, *Lipschitz constants for some approximation operators of a Lipschitz continuous function*, J. Approx. Theory, **59**(1989), no.3, 307-315.
- [19] Z. Li, *Bernstein polynomials and modulus of continuity*, J. Approx. Theory, **102**(2000), no.1, 171-174.
- [20] T. Lindvall, *Bernstein polynomials and the law of large numbers*, Math. Sci., **7**(1982), no.2, 127-139.
- [21] D. D. Stancu, C. Cismaşiu, *On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numér. Théor. Approx., **26**(1997), no.1-2, 221-227.
- [22] D. D. Stancu, L. A. Căbulea, D. Pop, *Approximation of bivariate functions by means of the operators  $S_{m,n}^{\alpha,\beta;a,b}$* , Stud. Univ. Babeş-Bolyai Math., **47**(2002), no.4, 105-113.
- [23] D. D. Stancu, *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Anal. Numér. Théor. Approx., **31**(2002), no.1, 115-118.
- [24] D. D. Stancu, E. I. Stoica, *On the use Abel-Jensen type combinatorial formulas for construction and investigation of some algebraic polynomial operators of approximation*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no.4, 167-182.
- [25] E. I. Stoica, *On the combinatorial identities of Abel-Hurwitz type and their use in constructive theory of functions*, Stud. Univ. Babeş-Bolyai Math., **55**(2010), no.4, 249-257.
- [26] I. Taşcu, *Approximation of bivariate functions by operators of Stancu-Hurwitz type*, Facta Univ. Ser. Math. Inform., (2005), no.20, 33-39.
- [27] I. Taşcu, A. Horvat-Marc, *Construction of Stancu-Hurwitz operator for two variables*, Acta Univ. Apulensis Math. Inform., (2006), no.11, 97-101.
- [28] T. Trif, *An elementary proof of the preservation of Lipschitz constants by the Meyer-König and Zeller operators*, J. Inequal. Pure Appl. Math., **4**(2003), no.5, Article90, 3pp.

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